

The Simple Criteria of SLOCC Equivalence Classes ¹

Dafa Li^{a2}, Xiangrong Li^b, Hongtao Huang^c, Xinxin Li^d

^a Dept of mathematical sciences, Tsinghua University, Beijing 100084 CHINA

^b Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

^c Electrical Engineering and Computer Science Department

University of Michigan, Ann Arbor, MI 48109, USA

^d Dept. of computer science, Wayne State University, Detroit, MI 48202, USA

Abstract

We put forward an alternative approach to the SLOCC classification of entanglement states of three-qubit and four-qubit systems. By directly solving matrix equations, we obtain the relations satisfied by the amplitudes of states. The relations are readily tested since in them only addition, subtraction and multiplication occur.

Keywords: entanglement, quantum computing, quantum information, separability, SLOCC.

PACS numbers: 03.65.Bz, 03.65.Hk

¹The paper was supported by NSFC(Grant No. 60433050), the basic research fund of Tsinghua university No: JC2003043 and partially by the state key lab. of intelligence technology and system

²email address:dli@math.tsinghua.edu.cn

1 Introduction

Entanglement plays a key role in quantum computing and quantum information theory. One of the interesting issues on entanglement is how to define the equivalence of two entangled states. If two states can be obtained from each other by means of local operations and classical communication (LOCC) with nonzero probability, we say that two states have the same kind of entanglement[1]. It is well known that a pure entangled state of two-qubits can be locally transformed into a EPR state. For multipartite systems, there are several inequivalent forms of entanglement under asymptotic LOCC[2]. Recently, many authors investigated the equivalence classes of three-qubit states specified by stochastic local operations and classical communication (SLOCC)[3]–[11]. Dür et al showed that for pure states of three-qubits there are total six different classes of the entanglement and out of the six classes, there are two inequivalent types of genuine tripartite entanglement[4]. They put forward a principled method to distinguish the six classes from each other by calculating the ranks of the reduced density matrices and the minimal product decomposition[4]. For example, they pointed out that if a state of a three-qubit system with $r(\rho_A) = r(\rho_B) = r(\rho_C) = 2$ has 2 (resp. 3) product terms in its minimal product decomposition under SLOCC, then the state is equivalent to $|GHZ\rangle$ (resp. $|W\rangle$). However, so far no criterion is proposed for the minimal number of product decomposition terms under SLOCC[3][4][12]. In a more recent paper, Verstraete et al[9] considered the entanglement class of four-qubits under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement. In these previous papers, the authors just put forward some principled rules of classifying the entangled states. It needs complicated calculations when these principled rules is used to real states. It will be quite useful if a feasible approach can be found. Here, we present an alternative approach to classify the entanglement of three-qubits, and then generalize the case of four-qubits. We will give simple criteria of distinguishing the entanglement classes from each other simply by checking the relations satisfied by the amplitudes of the states.

2 Classification of entanglement for a three-qubit system

We first discuss the system comprising three qubits A, B, C. The states of a three-qubit system can be generally expressed as

$$|\psi\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle.$$

Two states $|\psi\rangle$ and $|\psi'\rangle$, are equivalent under SLOCC if and only if there exist invertible local operators α, β and γ such that

$$|\psi\rangle = \alpha \otimes \beta \otimes \gamma |\psi'\rangle, \quad (1)$$

where the local operators α, β and γ can be expressed as 2×2 invertible matrices

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}.$$

We consider the following six classes, respectively.

2.1 The class equivalent to the state $|GHZ\rangle$

Let $|\psi'\rangle \equiv |GHZ\rangle$, i.e.

$$|\psi'\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (2)$$

Substituting Eq. (2) into Eq. (1), we get

$$\begin{aligned} a_0 &= (\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2)/\sqrt{2}, & a_1 &= (\alpha_1\beta_1\gamma_3 + \alpha_2\beta_2\gamma_4)/\sqrt{2}, \\ a_2 &= (\alpha_1\beta_3\gamma_1 + \alpha_2\beta_4\gamma_2)/\sqrt{2}, & a_3 &= (\alpha_1\beta_3\gamma_3 + \alpha_2\beta_4\gamma_4)/\sqrt{2}, \\ a_4 &= (\alpha_3\beta_1\gamma_1 + \alpha_4\beta_2\gamma_2)/\sqrt{2}, & a_5 &= (\alpha_3\beta_1\gamma_3 + \alpha_4\beta_2\gamma_4)/\sqrt{2}, \\ a_6 &= (\alpha_3\beta_3\gamma_1 + \alpha_4\beta_4\gamma_2)/\sqrt{2}, & a_7 &= (\alpha_3\beta_3\gamma_3 + \alpha_4\beta_4\gamma_4)/\sqrt{2}. \end{aligned}$$

By calculating $a_i a_j - a_k a_l$, where $i + j = k + l$ and $i \oplus j = k \oplus l$, where \oplus is addition modulo two, we obtain the following equations:

$$\begin{aligned} a_2 a_4 - a_0 a_6 &= \gamma_2 \gamma_1 (\alpha_1 \alpha_4 - \alpha_3 \alpha_2) (\beta_2 \beta_3 - \beta_4 \beta_1) / 2, \\ a_3 a_5 - a_1 a_7 &= \gamma_4 \gamma_3 (\alpha_1 \alpha_4 - \alpha_3 \alpha_2) (\beta_2 \beta_3 - \beta_4 \beta_1) / 2, \\ a_0 a_7 - a_3 a_4 &= -(\alpha_1 \alpha_4 - \alpha_2 \alpha_3) (-\gamma_4 \beta_4 \gamma_1 \beta_1 + \beta_3 \gamma_3 \beta_2 \gamma_2) / 2, \\ a_1 a_6 - a_2 a_5 &= -(-\alpha_3 \alpha_2 + \alpha_1 \alpha_4) (-\gamma_2 \beta_4 \gamma_3 \beta_1 + \beta_3 \gamma_1 \beta_2 \gamma_4) / 2. \end{aligned}$$

By using the above equations, we further obtain

$$\begin{aligned} & (a_0 a_7 - a_2 a_5 + a_1 a_6 - a_3 a_4)^2 - 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7) \\ &= \frac{1}{4} (\alpha_1 \alpha_4 - \alpha_2 \alpha_3)^2 (-\gamma_4 \gamma_1 + \gamma_3 \gamma_2)^2 (-\beta_4 \beta_1 + \beta_2 \beta_3)^2. \end{aligned} \quad (3)$$

$|\psi\rangle$ is equivalent to $|GHZ\rangle$ under SLOCC, if and only if the invertible operators α, β and γ exist. From Eq. (3), we may immediately conclude that the necessary and sufficient condition of $|\psi\rangle$ being equivalent to $|GHZ\rangle$ is

$$(a_0 a_7 - a_2 a_5 + a_1 a_6 - a_3 a_4)^2 - 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7) \neq 0. \quad (4)$$

It is not hard to verify that

$$\begin{aligned} & (a_0 a_7 - a_2 a_5 + (a_1 a_6 - a_3 a_4))^2 - 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7) = \\ & (a_0 a_7 - a_3 a_4 - (a_1 a_6 - a_2 a_5))^2 - 4(a_1 a_4 - a_0 a_5)(a_3 a_6 - a_2 a_7) = \\ & (a_0 a_7 - a_2 a_5 - (a_1 a_6 - a_3 a_4))^2 - 4(a_0 a_3 - a_1 a_2)(a_4 a_7 - a_5 a_6). \end{aligned} \quad (5)$$

Therefore the above condition (??) can be replaced by the following any one of the following conditions.

$$\begin{aligned} (a_0a_7 - a_3a_4 - (a_1a_6 - a_2a_5))^2 - 4(a_1a_4 - a_0a_5)(a_3a_6 - a_2a_7) &\neq 0, \\ (a_0a_7 - a_2a_5 - (a_1a_6 - a_3a_4))^2 - 4(a_0a_3 - a_1a_2)(a_4a_7 - a_5a_6) &\neq 0. \end{aligned} \quad (6)$$

2.2 The class equivalent to the state $|W\rangle$

Let $|\psi'\rangle \equiv |W\rangle$, i.e.

$$|\psi'\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \quad (7)$$

Substituting Eq. (7) into Eq. (1), we get

$$\begin{aligned} a_0 &= (\alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_1)/\sqrt{3}, & a_1 &= (\alpha_1\beta_1\gamma_4 + \alpha_1\beta_2\gamma_3 + \alpha_2\beta_1\gamma_3)/\sqrt{3}, \\ a_2 &= (\alpha_1\beta_3\gamma_2 + \alpha_1\beta_4\gamma_1 + \alpha_2\beta_3\gamma_1)/\sqrt{3}, & a_3 &= (\alpha_1\beta_3\gamma_4 + \alpha_1\beta_4\gamma_3 + \alpha_2\beta_3\gamma_3)/\sqrt{3}, \\ a_4 &= (\alpha_3\beta_1\gamma_2 + \alpha_3\beta_2\gamma_1 + \alpha_4\beta_1\gamma_1)/\sqrt{3}, & a_5 &= (\alpha_3\beta_1\gamma_4 + \alpha_3\beta_2\gamma_3 + \alpha_4\beta_1\gamma_3)/\sqrt{3}, \\ a_6 &= (\alpha_3\beta_3\gamma_2 + \alpha_3\beta_4\gamma_1 + \alpha_4\beta_3\gamma_1)/\sqrt{3}, & a_7 &= (\alpha_3\beta_3\gamma_4 + \alpha_3\beta_4\gamma_3 + \alpha_4\beta_3\gamma_3)/\sqrt{3}. \end{aligned} \quad (8)$$

By calculating $a_ia_j - a_ka_l$, where $i + j = k + l$ and $i \oplus j = k \oplus l$, we have

$$\begin{aligned} a_0a_3 - a_1a_2 &= \alpha_1^2(-\beta_2\beta_3 + \beta_1\beta_4)(\gamma_2\gamma_3 - \gamma_4\gamma_1)/3, \\ a_5a_6 - a_4a_7 &= -\alpha_3^2(-\beta_2\beta_3 + \beta_1\beta_4)(\gamma_2\gamma_3 - \gamma_4\gamma_1)/3, \\ a_1a_4 - a_0a_5 &= -\beta_1^2(\gamma_2\gamma_3 - \gamma_4\gamma_1)(\alpha_1\alpha_4 - \alpha_3\alpha_2)/3, \\ a_3a_6 - a_2a_7 &= -\beta_3^2(\gamma_2\gamma_3 - \gamma_4\gamma_1)(\alpha_1\alpha_4 - \alpha_3\alpha_2)/3, \\ a_3a_5 - a_1a_7 &= \gamma_3^2(-\beta_2\beta_3 + \beta_1\beta_4)(\alpha_1\alpha_4 - \alpha_3\alpha_2)/3, \\ a_2a_4 - a_0a_6 &= \gamma_1^2(-\beta_2\beta_3 + \beta_1\beta_4)(\alpha_1\alpha_4 - \alpha_3\alpha_2)/3, \\ a_0a_7 - a_3a_4 &= (\alpha_1\alpha_4 - \alpha_2\alpha_3)(\gamma_1\gamma_3(\beta_2\beta_3 - \beta_1\beta_4) + \beta_1\beta_3(\gamma_2\gamma_3 - \gamma_1\gamma_4))/3, \\ a_1a_6 - a_2a_5 &= -(\alpha_1\alpha_4 - \alpha_2\alpha_3)(\gamma_1\gamma_3(\beta_1\beta_4 - \beta_2\beta_3) + \beta_1\beta_3(\gamma_2\gamma_3 - \gamma_1\gamma_4))/3, \\ a_0a_7 - a_2a_5 &= (\beta_1\beta_4 - \beta_2\beta_3)(\alpha_1\alpha_3(\gamma_2\gamma_3 - \gamma_1\gamma_4) + \gamma_1\gamma_3(\alpha_2\alpha_3 - \alpha_1\alpha_4))/3, \\ a_1a_6 - a_3a_4 &= -(\beta_1\beta_4 - \beta_2\beta_3)(\alpha_1\alpha_3(\gamma_2\gamma_3 - \gamma_1\gamma_4) + \gamma_1\gamma_3(\alpha_1\alpha_4 - \alpha_2\alpha_3))/3. \end{aligned}$$

By using the above equations, we can conclude that $|\psi\rangle$ is equivalent to $|W\rangle$ under SLOCC if and only if a_i satisfy the following equation

$$(a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4))^2 - 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) = 0 \quad (9)$$

and inequalities

$$\begin{aligned} a_0a_3 &\neq a_1a_2 \vee a_5a_6 \neq a_4a_7, \\ a_1a_4 &\neq a_0a_5 \vee a_3a_6 \neq a_2a_7, \\ a_3a_5 &\neq a_1a_7 \vee a_2a_4 \neq a_0a_6. \end{aligned} \quad (10)$$

Notice that from (5) Eq. (9) can be replaced by any one of the following equations.

$$\begin{aligned} (a_0a_7 - a_3a_4 - (a_1a_6 - a_2a_5))^2 - 4(a_1a_4 - a_0a_5)(a_3a_6 - a_2a_7) &= 0, \\ (a_0a_7 - a_2a_5 - (a_1a_6 - a_3a_4))^2 - 4(a_0a_3 - a_1a_2)(a_4a_7 - a_5a_6) &= 0. \end{aligned} \quad (11)$$

2.3 A-BC class

If $|\psi\rangle$ belongs to A-BC class, then $|\psi\rangle$ can be written as $|\psi\rangle = (s|0\rangle_A + t|1\rangle_A)(a|00\rangle_{BC} + b|01\rangle_{BC} + c|10\rangle_{BC} + d|11\rangle_{BC})$, where $bc \neq ad$ since systems B and C are entangled. Thus we obtain the following equations.

$$\begin{aligned} as &= a_0, & bs &= a_1, & cs &= a_2, & ds &= a_3, \\ at &= a_4, & bt &= a_5, & ct &= a_6, & dt &= a_7. \end{aligned} \quad (12)$$

By using the equations (12) and $bc \neq ad$, we can obtain the following equalities and inequalities,

$$\begin{aligned} a_0a_5 &= a_1a_4, & a_2a_7 &= a_3a_6, \\ a_0a_6 &= a_2a_4, & a_1a_7 &= a_3a_5, \\ a_0a_7 &= a_3a_4, & a_1a_6 &= a_2a_5; \\ a_1a_2 &\neq a_0a_3 \vee a_5a_6 \neq a_4a_7. \end{aligned} \quad (13)$$

It is clear that the relations in Eq. (13) are the necessary condition of $|\psi\rangle$ being equivalent to the class A-BC. Conversely, we can show that this criterion is sufficient too. To this end, let $|s|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$, $|t|^2 = |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2$, $|a|^2 = |a_0|^2 + |a_4|^2$, $|b|^2 = |a_1|^2 + |a_5|^2$, $|c|^2 = |a_2|^2 + |a_6|^2$ and $|d|^2 = |a_3|^2 + |a_7|^2$. For the real number case, we can see that the above amplitude equations in (12) hold true under the equalities in (13). For example, $a^2s^2 = (a_0^2 + a_4^2)(a_0^2 + a_1^2 + a_2^2 + a_3^2) = a_0^2(a_0^2 + a_1^2 + a_2^2 + a_3^2) + a_0^2a_4^2 + a_1^2a_4^2 + a_2^2a_4^2 + a_3^2a_4^2 = a_0^2(a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2) = a_0^2$. These amplitude equations imply that $|\psi\rangle = (s|0\rangle_A + t|1\rangle_A)(a|00\rangle_{BC} + b|01\rangle_{BC} + c|10\rangle_{BC} + d|11\rangle_{BC})$. This case can be extended to the complex case.

Next we show that systems B and C are entangled. From the amplitude equations (12) we can obtain that $bcs^2 = a_1a_2$, $ads^2 = a_0a_3$, $bct^2 = a_5a_6$ and $adt^2 = a_4a_7$. Further from the inequalities of this criterion, it is easy to derive that $bcs^2 \neq ads^2$ or $bct^2 \neq adt^2$. Since at least one of s and t is not zero, then $bc \neq ad$, which means that systems B and C are entangled.

We arrive at the conclusion that $|\psi\rangle$ belongs to the A-BC class if and only if a_i satisfy the equalities and inequalities given in Eq. (13).

Notice that it can be verified by using (12) that (9) also holds for $A - BC$ class.

2.4 B-AC class

Similarly, we can show that $|\psi\rangle$ belongs to this class if and only if a_i satisfy the following equalities and inequalities:

$$\begin{aligned} a_0a_3 &= a_1a_2, & a_4a_7 &= a_5a_6, \\ a_0a_6 &= a_2a_4, & a_1a_7 &= a_3a_5, \\ a_0a_7 &= a_2a_5 & a_1a_6 &= a_3a_4; \\ a_1a_4 &\neq a_0a_5 \vee a_3a_6 \neq a_2a_7. \end{aligned} \tag{14}$$

The proof is similar to the one for class $A - BC$.

Notice that (9) also holds for $B - AC$ class.

2.5 C-AB class

$|\psi\rangle$ belongs to the class C-AB, if and only if a_i satisfy the following equalities and inequalities:

$$\begin{aligned} a_0a_3 &= a_1a_2, & a_4a_7 &= a_5a_6, \\ a_0a_5 &= a_1a_4, & a_2a_7 &= a_3a_6, \\ a_0a_7 &= a_1a_6, & a_2a_5 &= a_3a_4; \\ a_2a_4 &\neq a_0a_6 \vee a_3a_5 \neq a_1a_7. \end{aligned} \tag{15}$$

The proof is similar to the one for class $A - BC$.

Notice that (9) also holds for $C - AB$ class.

2.6 A - B - C class

If the state $|\psi\rangle$ belongs to the class A-B-C, the necessary and sufficient condition[13] reads

$$\begin{aligned} a_0a_3 &= a_1a_2, & a_5a_6 &= a_4a_7, \\ a_0a_5 &= a_1a_4, & a_3a_6 &= a_2a_7, \\ a_1a_7 &= a_3a_5, & a_2a_4 &= a_0a_6, \\ a_0a_7 &= a_1a_6, & a_2a_5 &= a_3a_4. \end{aligned} \tag{16}$$

Notice that it is easy to see from (16) that (9) holds.

2.7 The complete partition

Before proceeding further, we would like to point out that the criteria for classes $A - B - C$, $A - BC$, $B - AC$, $C - AB$, $|GHZ\rangle$ and $|W\rangle$ are exclusive to each other (see Appendix A for the details). The criteria form a complete partition (see table 1 for the details). For this completeness, what we need to do next is to demonstrate the “not-occur” cases and the alternate criteria for $A - BC$, $B - AC$, $C - AB$, $A - B - C$ in table 1. We finish these proofs in appendixes B,

C and D respectively. Thus we argue that any state has to be in one of classes $A - B - C$, $A - BC$, $B - AC$, $C - AB$, $|GHZ\rangle$ and $|W\rangle$ and any two states satisfying the same criterion are related by SLOCC. In other words, we give a different proof of Dür et al.'s SLOCC classification.

The table 1: The complete partition

$(a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4))^2$ $= 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7)$	$a_0a_3 = a_1a_2$ $\wedge a_5a_6 = a_4a_7$	$a_1a_4 = a_0a_5$ $\wedge a_3a_6 = a_2a_7$	$a_3a_5 = a_1a_7$ $\wedge a_2a_4 = a_0a_6$	
N	$ GHZ\rangle$			
Y	N	N	N	$ W\rangle$
		Y	Y	not-occur
			N	not-occur
		Y	Y	$A - BC$
	Y	N	N	not-occur
			Y	$B - AC$
		Y	N	$C - AB$
			Y	$A - B - C$

In the table 1, “Y” means that the condition holds and “N” means that the condition does not hold. “not-occur” means the case does not occur.

3 Classification of entanglement for a four-qubit system

We now turn the discussion to four-qubit systems. By means of the criteria for SLOCC entanglement classes of three-qubits, we can derive the criteria for degenerated four-qubit entanglement. We give the necessary criteria which the four-qubit $|GHZ\rangle$ and $|W\rangle$ classes satisfy. Let $|C_4\rangle = (|0011\rangle + |0110\rangle + |1100\rangle + |1010\rangle + |1001\rangle + |0101\rangle)/\sqrt{6}$. We argue that $|C_4\rangle$ is a genuinely four-qubit entangled state which is inequivalent to $|GHZ\rangle$ or $|W\rangle$ states of four-qubits under SLOCC.

Let $|\psi\rangle = \sum_{j=0}^{15} a_j|j\rangle$ be any pure state of four-qubits.

3.1 Three-qubit GHZ entanglement accompanied with a separable one qubit

We only study that ABC are GHZ -entangled. Let $|\psi\rangle = |\varphi\rangle_{ABC}(s|0\rangle + t|1\rangle)_D$, where $|\varphi\rangle = \sum_{i=0}^7 b_i|i\rangle$ and $|\varphi\rangle$ is in $|GHZ\rangle$ class of three-qubits. By the criterion for $|GHZ\rangle$ class of three-qubits, we have the following inequality,

$$(b_0b_7 - b_2b_5 + (b_1b_6 - b_3b_4))^2 \neq 4(b_2b_4 - b_0b_6)(b_3b_5 - b_1b_7).$$

Since $\gamma \neq 0$ or $\delta \neq 0$,

$$[(b_0b_7 - b_2b_5 + (b_1b_6 - b_3b_4))^2 - 4(b_2b_4 - b_0b_6)(b_3b_5 - b_1b_7)]\gamma^4 \neq 0 \text{ or}$$

$$[(b_0b_7 - b_2b_5 + (b_1b_6 - b_3b_4))^2 - 4(b_2b_4 - b_0b_6)(b_3b_5 - b_1b_7)]\delta^4 \neq 0.$$

Consequently, a_i satisfy the following inequalities:

$$(a_0a_{14} - a_4a_{10} + a_2a_{12} - a_6a_8)^2 \neq 4(a_4a_8 - a_0a_{12})(a_6a_{10} - a_2a_{14}) \text{ or}$$

$$(a_1a_{15} - a_5a_{11} + a_3a_{13} - a_7a_9)^2 \neq 4(a_5a_9 - a_1a_{13})(a_7a_{11} - a_3a_{15})$$

and the following equalities:

$a_i a_j = a_k a_l$, where $a_i a_j - a_k a_l$ are all the 2×2 minor determinants of the following matrix,

$$\begin{pmatrix} a_0 & a_2 & a_4 & a_6 & a_8 & a_{10} & a_{12} & a_{14} \\ a_1 & a_3 & a_5 & a_7 & a_9 & a_{11} & a_{13} & a_{15} \end{pmatrix}.$$

These conditions are necessary and sufficient.

Example, $(|0000\rangle + |1110\rangle)/\sqrt{2}$ satisfies the above conditions.

3.2 Three-qubit W entanglement accompanied with a separable one qubit

We only illustrate that ABC are W -entangled. Let $|\psi\rangle = |\varphi\rangle_{ABC}(s|0\rangle + t|1\rangle)_D$, where $|\varphi\rangle = \sum_{i=0}^7 b_i |i\rangle$ and $|\varphi\rangle$ is in $|W\rangle$ class of three-qubits. By means of the criterion for $|W\rangle$ class of three-qubits, a_i satisfy the following equalities:

$$(a_0 a_{14} - a_4 a_{10} + a_2 a_{12} - a_6 a_8)^2 = 4(a_4 a_8 - a_0 a_{12})(a_6 a_{10} - a_2 a_{14}),$$

$$(a_1 a_{15} - a_5 a_{11} + a_3 a_{13} - a_7 a_9)^2 = 4(a_5 a_9 - a_1 a_{13})(a_7 a_{11} - a_3 a_{15}),$$

$a_i a_j = a_k a_l$, where $a_i a_j - a_k a_l$ are all the 2×2 minor determinants of the following matrix,

$$\begin{pmatrix} a_0 & a_2 & a_4 & a_6 & a_8 & a_{10} & a_{12} & a_{14} \\ a_1 & a_3 & a_5 & a_7 & a_9 & a_{11} & a_{13} & a_{15} \end{pmatrix}$$

and satisfy the following inequalities:

$$(a_3 a_5 \neq a_1 a_7 \vee a_{10} a_{12} \neq a_8 a_{14} \vee a_2 a_4 \neq a_0 a_6 \vee a_{11} a_{13} \neq a_9 a_{15}) \wedge$$

$$(a_2 a_8 \neq a_0 a_{10} \vee a_3 a_9 \neq a_1 a_{11} \vee a_6 a_{12} \neq a_4 a_{14} \vee a_7 a_{13} \neq a_5 a_{15}) \wedge$$

$$(a_6 a_{10} \neq a_2 a_{14} \vee a_7 a_{11} \neq a_3 a_{15} \vee a_4 a_8 \neq a_0 a_{12} \vee a_5 a_9 \neq a_1 a_{13}).$$

These conditions are necessary and sufficient.

For example, $|W\rangle_{123} \otimes |0\rangle_4$ and $(|110\rangle_{123} + |101\rangle_{123} + |011\rangle_{123}) \otimes |0\rangle_4$ satisfy the above conditions.

3.3 A state consisting of two EPR pairs

We only investigate $AB - CD$ class, where AB and CD are EPR pairs, as follows. $|\psi\rangle$ is in this class if and only if a_i satisfy the following inequalities

$$(a_4 a_8 \neq a_0 a_{12} \vee a_6 a_{10} \neq a_2 a_{14} \vee a_5 a_9 \neq a_1 a_{13} \vee a_7 a_{11} \neq a_3 a_{15}) \wedge$$

$$(a_1 a_2 \neq a_0 a_3 \vee a_5 a_6 \neq a_4 a_7 \vee a_9 a_{10} \neq a_8 a_{11} \vee a_{13} a_{14} \neq a_{12} a_{15})$$

and the following equalities:

$a_i a_j = a_k a_l$, where $a_i a_j - a_k a_l$ are all the 2×2 minor determinants of the following matrix,

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}.$$

Example, $|\phi_4\rangle$, which is $(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$ in [6], does not satisfy the above conditions.

3.4 Only two qubits are entangled

We only discuss $A - B - CD$ class where CD is a EPR pair. Then one can obtain that $|\psi\rangle$ is in this class if and only if a_i satisfy the following inequalities

$$a_1a_2 \neq a_0a_3 \vee a_5a_6 \neq a_4a_7 \vee a_9a_{10} \neq a_8a_{11} \vee a_{13}a_{14} \neq a_{12}a_{15}$$

and the following equalities

$$a_ia_j = a_ka_l, \text{ where } i+j = k+l \text{ and } i \oplus j = k \oplus l, i < j, k < l, i = l \pmod{4}, j = k \pmod{4}.$$

3.5 A-B-C-D class

$|\psi\rangle$ is separable if and only if $a_ia_j = a_ka_l$, where $i+j = k+l$ and $i \oplus j = k \oplus l$.

3.6 $|GHZ\rangle$ class

Let $|GHZ\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$. Then if $|\psi\rangle$ is equivalent to $|GHZ\rangle$ under SLOCC then a_i satisfy the following inequality,

$$a_2a_{13} - a_3a_{12} + (a_4a_{11} - a_5a_{10}) \neq (a_0a_{15} - a_1a_{14}) + (a_6a_9 - a_7a_8)$$

and the following equalities,

$$(a_1a_4 - a_0a_5)(a_{11}a_{14} - a_{10}a_{15}) = (a_3a_6 - a_2a_7)(a_9a_{12} - a_8a_{13}),$$

$$(a_4a_7 - a_5a_6)(a_8a_{11} - a_9a_{10}) = (a_0a_3 - a_1a_2)(a_{12}a_{15} - a_{13}a_{14}),$$

$$(a_3a_5 - a_1a_7)(a_{10}a_{12} - a_8a_{14}) = (a_2a_4 - a_0a_6)(a_{11}a_{13} - a_9a_{15}).$$

For example, $|GHZ\rangle$ satisfies the above conditions, while $|\phi_4\rangle$ [6] does not satisfy the second equality of this criterion. Thus, we also show that $|\phi_4\rangle$ is not in the four-qubit GHZ entanglement class.

3.7 $|W\rangle$ class

Let $|W\rangle = (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/2$. Then if $|\psi\rangle$ is equivalent to $|W\rangle$ under SLOCC then a_i satisfy the following equalities

$$\begin{aligned} a_2a_{13} - a_3a_{12} + a_4a_{11} - a_5a_{10} &= a_0a_{15} - a_1a_{14} + a_6a_9 - a_7a_8 \\ ((a_0a_7 - a_3a_4) + (a_1a_6 - a_2a_5))^2 &= 4(a_3a_5 - a_1a_7)(a_2a_4 - a_0a_6), \\ ((a_4a_{11} - a_7a_8) + (a_5a_{10} - a_6a_9))^2 &= 4(a_7a_9 - a_5a_{11})(a_6a_8 - a_4a_{10}), \\ ((a_8a_{15} - a_{11}a_{12}) + (a_9a_{14} - a_{10}a_{13}))^2 &= 4(a_{11}a_{13} - a_9a_{15})(a_{10}a_{12} - a_8a_{14}), \\ (a_0a_{14} - a_4a_{10} + a_2a_{12} - a_6a_8)^2 &= 4(a_4a_8 - a_0a_{12})(a_6a_{10} - a_2a_{14}) \\ (a_1a_{15} - a_5a_{11} + a_3a_{13} - a_7a_9)^2 &= 4(a_5a_9 - a_1a_{13})(a_7a_{11} - a_3a_{15}) \\ (a_0a_{11} - a_2a_9 + a_1a_{10} - a_3a_8)^2 &= 4(a_2a_8 - a_0a_{10})(a_3a_9 - a_1a_{11}) \\ (a_4a_{15} - a_6a_{13} + a_5a_{14} - a_7a_{12})^2 &= 4(a_6a_{12} - a_4a_{14})(a_7a_{13} - a_5a_{15}) \\ (a_0a_{13} - a_4a_9 + a_1a_{12} - a_5a_8)^2 &= 4(a_4a_8 - a_0a_{12})(a_5a_9 - a_1a_{13}) \\ (a_2a_{15} - a_6a_{11} + a_3a_{14} - a_7a_{10})^2 &= 4(a_6a_{10} - a_2a_{14})(a_7a_{11} - a_3a_{15}) \\ (a_1a_4 - a_0a_5)(a_{11}a_{14} - a_{10}a_{15}) &= (a_3a_6 - a_2a_7)(a_9a_{12} - a_8a_{13}), \\ (a_4a_7 - a_5a_6)(a_8a_{11} - a_9a_{10}) &= (a_0a_3 - a_1a_2)(a_{12}a_{15} - a_{13}a_{14}), \\ (a_3a_5 - a_1a_7)(a_{10}a_{12} - a_8a_{14}) &= (a_2a_4 - a_0a_6)(a_{11}a_{13} - a_9a_{15}), \end{aligned}$$

and the following inequalities

$$\begin{aligned} a_0a_3 \neq a_1a_2 \text{ or } a_5a_6 \neq a_4a_7 \text{ or } a_8a_{11} \neq a_9a_{10} \text{ or } a_{13}a_{14} \neq a_{12}a_{15}, \\ a_1a_4 \neq a_0a_5 \text{ or } a_3a_6 \neq a_2a_7 \text{ or } a_9a_{12} \neq a_8a_{13} \text{ or } a_{11}a_{14} \neq a_{10}a_{15}, \end{aligned}$$

and $a_3a_5 \neq a_1a_7$ or $a_2a_4 \neq a_0a_6$ or $a_{11}a_{13} \neq a_9a_{15}$ or $a_{10}a_{12} \neq a_8a_{14}$.

The proof is in appendix E.

Example, $|W\rangle$ satisfies the above conditions, while $|\phi_4\rangle$ does not satisfy this criterion.

3.8 A genuinely four-qubit entanglement $|C_4\rangle$ class

It is easy to verify that $|C_4\rangle$ does not satisfy the criteria for degenerated four-qubit entanglement. It means that $|C_4\rangle$ is a genuinely four-qubit entangled state. We can also observe that $|C_4\rangle$ does not satisfy the first equality of the criteria for $|GHZ\rangle$ or $|W\rangle$ classes. Therefore, $|C_4\rangle$ is inequivalent to $|GHZ\rangle$ or $|W\rangle$ under SLOCC.

$|C_4\rangle$ possesses the following properties.

(1). $|C_4\rangle$ is symmetric under permutation of parties.

(2). $|C_4\rangle$ is self-complementary in the following sense.

Let $\bar{1}$ ($\bar{0}$) be the complement of a bit 1 (0). Then $\bar{\bar{0}} = 1$ and $\bar{\bar{1}} = 0$. Let $\bar{z} = \bar{z}_1\bar{z}_2\dots\bar{z}_n$ denote the complement of a binary string $z = z_1z_2\dots z_n$. Likewise, we can define $|\bar{\Phi}\rangle = c_0|\bar{0}\rangle + c_1|\bar{1}\rangle + \dots + c_{2^n-1}|\bar{(2^n-1)}\rangle$, where $|\Phi\rangle = c_0|0\rangle + c_1|1\rangle + \dots + c_{2^n-1}|2^n-1\rangle$, and call $|\bar{\Phi}\rangle$ the complement of $|\Phi\rangle$. Obviously, $|C_4\rangle = |\bar{C}_4\rangle$.

(3). When any one of the four qubits is traced out, the remaining three qubits are identical and mixed. For example, $tr_D(|C_4\rangle\langle C_4|) = (|W\rangle\langle W| + |\bar{W}\rangle\langle \bar{W}|)/2$, where $|W\rangle$ is $(|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$.

(4). When any two of the four qubits are traced out, the remaining two qubits are identical and mixed. For example, $tr_{CD}(|C_4\rangle\langle C_4|) = \frac{1}{6}(|11\rangle\langle 11| + |00\rangle\langle 00|) + \frac{2}{3}|\Psi^+\rangle\langle \Psi^+|$, where $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$.

4 Summaries

In summaries, in this paper we propose the simple criteria, in which only addition and multiplication occur, for the SLOCC equivalence classes and show that these criteria are exclusive and form a complete partition. Thus, new proofs are given for Dür et al.'s SLOCC classification of three-qubits. By means of the criteria for SLOCC entanglement classes of three-qubits, we can derive the criteria for degenerated four-qubit entanglement. We obtain the necessary criteria which the four-qubit $|GHZ\rangle$ and $|W\rangle$ classes satisfy. We observe that $|C_4\rangle$ is a genuinely four-qubit entangled state which is inequivalent to $|GHZ\rangle$ or $|W\rangle$ states of four-qubits under SLOCC. By means of the criteria of four-qubits, we can determine if a state is a genuinely four-qubit entangled state which is inequivalent to $|GHZ\rangle$ or $|W\rangle$ under SLOCC.

Appendix A

Proof.

Case 1. The criteria for $|GHZ\rangle$ and $|W\rangle$ classes are exclusive.

This is because the criterion for $|GHZ\rangle$ class contradicts condition (1) of the criterion for $|W\rangle$ class.

Case 2. The criteria for class $|W\rangle$ and for any one of classes $A - B - C$, $A - BC$, $B - AC$ and $C - AB$ are exclusive.

Condition (2) of the criterion for $|W\rangle$ class implies that the criteria for $|W\rangle$ class and $A - B - C$ class are exclusive.

In $A - BC$ class, any state satisfies $a_1a_4 = a_0a_5$ and $a_3a_6 = a_2a_7$. However, condition (2) of the criterion for $|W\rangle$ class says that $a_1a_4 \neq a_0a_5$ or $a_3a_6 \neq a_2a_7$. Therefore the criteria for $|W\rangle$ class and $A - BC$ class are exclusive.

Similarly, the criteria for classes $|W\rangle$ and $B - AC$ and for classes $|W\rangle$ and $C - AB$ are exclusive, respectively.

Case 3. The criteria for class $|GHZ\rangle$ and for any one of classes $A - B - C$, $A - BC$, $B - AC$ and $C - AB$ are exclusive.

First let us demonstrate that the criteria for classes $|GHZ\rangle$ and $A - B - C$ are exclusive. We see immediately that any state in $A - B - C$ class does not satisfy the criteria for $|GHZ\rangle$ class. Conversely, if a state in $|GHZ\rangle$ class satisfies $a_3a_5 \neq a_1a_7$ or $a_2a_4 \neq a_0a_6$, then the state is not separable by the criterion for class $A - B - C$. Otherwise the state in $|GHZ\rangle$ class satisfies $a_3a_5 = a_1a_7$ and $a_2a_4 = a_0a_6$. By the criterion for $|GHZ\rangle$ class, it follows that $a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4) \neq 0$. It means that it is impossible that $a_0a_7 = a_2a_5$ and $a_1a_6 = a_3a_4$. Thus the state in $|GHZ\rangle$ class is not in class $A - B - C$. Therefore the criteria for classes $|GHZ\rangle$ and $A - B - C$ are exclusive.

Next let us deduce that the criteria for classes $|GHZ\rangle$ and $A - BC$ are exclusive. From section 2 we know that any state in $A - BC$ class satisfies $a_1a_4 = a_0a_5$, $a_3a_6 = a_2a_7$, $a_0a_7 = a_3a_4$ and $a_1a_6 = a_2a_5$. Using these equalities we can derive that $a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4) = 0$ and $(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) = 0$. It means that any state in $A - BC$ class does not satisfy criterion for $|GHZ\rangle$ class. Conversely, if a state in $|GHZ\rangle$ class satisfies $a_0a_7 \neq a_3a_4$ or $a_1a_6 \neq a_2a_5$, then the state in $|GHZ\rangle$ class is not in $A - BC$ class by the criterion for class $A - BC$. Otherwise the state in $|GHZ\rangle$ class satisfies $a_0a_7 = a_3a_4$ and $a_1a_6 = a_2a_5$. Then by the criterion for $|GHZ\rangle$ class, it follows that $(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) \neq 0$. This results in that the state in $|GHZ\rangle$ class is not in $A - BC$ class. Therefore the criteria for classes $|GHZ\rangle$ and $A - BC$ are exclusive.

Similarly, we can infer that the criteria for classes $|GHZ\rangle$ and $B - AC$, and for classes $|GHZ\rangle$ and $C - AB$ are exclusive, respectively.

Case 4. It is easy to see that the criteria for classes $A - B - C$, $A - BC$, $B - AC$ and $C - AB$ are pairwise exclusive.

Appendix B

Lemma. No states satisfy the following conditions. In other words, the conditions (1), (2), (3), (4) and (5) are inconsistent. Therefore the situation is not applicable. For other “not-occur” cases in table 1, the discussions are similar therefore omitted.

$$(a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4))^2 = 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7).....(1)$$

$$a_3a_5 = a_1a_7.....(2)$$

$$a_2a_4 = a_0a_6(3)$$

$$a_0a_3 \neq a_1a_2 \text{ or } a_5a_6 \neq a_4a_7(4)$$

$$a_1a_4 \neq a_0a_5 \text{ or } a_3a_6 \neq a_2a_7.....(5)$$

Proof.

From (1), (2) and (3), we have the following

$$a_0a_7 - a_3a_4 = a_2a_5 - a_1a_6 \dots\dots\dots(6).$$

Next we prove that (4) holds means that (5) does not hold under (1), (2) and (3). That is, $a_0a_3 \neq a_1a_2$ or $a_5a_6 \neq a_4a_7$ results in $a_1a_4 = a_0a_5$ and $a_3a_6 = a_2a_7$. There are two cases.

Case 1. $a_0a_3 \neq a_1a_2$ implies $a_1a_4 = a_0a_5$ and $a_3a_6 = a_2a_7$.

Case 1.1. Replacing a_1a_7 by a_3a_5 and a_0a_6 by a_2a_4 in $a_0a_1(a_0a_7 - a_3a_4) = a_0a_1(a_2a_5 - a_1a_6)$ from (6) respectively, and by factoring we have $(a_0a_3 - a_1a_2)(a_1a_4 - a_0a_5) = 0$. Therefore $a_0a_3 \neq a_1a_2$ yields $a_1a_4 = a_0a_5$.

Case 1.2. Replacing a_3a_5 by a_1a_7 and a_2a_4 by a_0a_6 in $a_2a_3(a_0a_7 - a_3a_4) = a_2a_3(a_2a_5 - a_1a_6)$ from (6) respectively, and by factoring we have

$$(a_0a_3 - a_1a_2)(a_2a_7 - a_3a_6) = 0, \text{ which yields } a_3a_6 = a_2a_7 \text{ since } a_0a_3 \neq a_1a_2.$$

Case 2. Suppose $a_5a_6 \neq a_4a_7$. Similarly we can derive $a_1a_4 = a_0a_5$ and $a_3a_6 = a_2a_7$.

Appendix C

Lemma. the criterion for $B - AC$ class is equivalent to the following six conditions.

$$a_0a_7 - a_3a_4 = a_2a_5 - a_1a_6 \dots\dots(1),$$

$$a_2a_4 = a_0a_6 \dots\dots(2),$$

$$a_3a_5 = a_1a_7 \dots\dots(3),$$

$$a_0a_3 = a_1a_2 \dots\dots(4),$$

$$a_5a_6 = a_4a_7 \dots\dots(5),$$

$$a_1a_4 \neq a_0a_5 \text{ or } a_3a_6 \neq a_2a_7 \dots\dots(6).$$

For $A - BC$ and $C - AB$, the discussion is similar to this one.

Proof. It is trivial to verify that the criterion for $B - AC$ in section 2 satisfies the above conditions. Conversely, we can prove that the above conditions satisfy the criterion for $B - AC$. It is enough to show $a_0a_7 = a_2a_5$ and $a_1a_6 = a_3a_4$. Replacing a_2a_4 by a_0a_6 and a_1a_2 by a_0a_3 respectively, in $a_2(a_0a_7 - a_3a_4) = a_2(a_2a_5 - a_1a_6)$ from (1), we obtain $a_0a_2a_7 = a_2^2a_5$. When $a_2 \neq 0$, we have $a_0a_7 = a_2a_5$. Otherwise it is straightforward to obtain $a_0a_7 = a_2a_5$ since it is trivial when $a_0 = 0$ and $a_0 \neq 0$ leads to $a_3 = a_6 = 0$ from (2) and (4) and further, $a_0a_7 = 0$ from (1). Similarly, replacing a_1a_7 by a_3a_5 and a_1a_2 by a_0a_3 respectively, in $a_1(a_0a_7 - a_3a_4) = a_1(a_2a_5 - a_1a_6)$ from (1), we get $a_1a_3a_4 = a_1^2a_6$. Then we observe $a_1a_6 = a_3a_4$ when $a_1 \neq 0$. Otherwise it is easy to find $a_1a_6 = a_3a_4$.

Appendix D

Lemma. The criterion for $A - B - C$ in section 2 is equivalent to the following equalities.

$$a_0a_7 - a_3a_4 = a_2a_5 - a_1a_6, a_2a_4 = a_0a_6, a_3a_5 = a_1a_7,$$

$$a_0a_3 = a_1a_2, a_5a_6 = a_4a_7, a_1a_4 = a_0a_5, a_3a_6 = a_2a_7.$$

Proof. It is easy to verify that the criterion for $A - B - C$ satisfies the above equalities. Conversely, in appendix C, by using the first five equalities we derive $a_0a_7 = a_2a_5$ and $a_1a_6 = a_3a_4$. Similarly, we can obtain $a_0a_7 = a_1a_6$, $a_1a_6 = a_2a_5$, $a_2a_5 = a_3a_4$, $a_0a_7 = a_3a_4$. Therefore the criterion for $A - B - C$ is satisfied.

Appendix E

The necessary criterion of the entanglement class $|W\rangle$ for a four-qubit system Proof.

Let α, β, γ and δ be operators and $|\psi\rangle = \sum_{i=0}^{15} a_i|i\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta|W\rangle$, where $\delta = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}$. Then

$a_0 = (\alpha_1\beta_1\gamma_1\delta_2 + \alpha_1\beta_1\gamma_2\delta_1 + \alpha_1\beta_2\gamma_1\delta_1 + \alpha_2\beta_1\gamma_1\delta_1)/2$ and other a_i are omitted.

Computing $a_i a_j - a_k a_l$, where $i + j = k + l$ and $i \oplus j = k \oplus l$, we obtain the following equations:

$$\begin{aligned} a_0 a_3 - a_1 a_2 &= \frac{1}{4} \alpha_1^2 \beta_1^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_4 a_7 - a_5 a_6 &= \frac{1}{4} \alpha_1^2 \beta_3^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_1 a_4 - a_0 a_5 &= \frac{1}{4} \alpha_1^2 \gamma_1^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (\beta_1 \beta_4 - \beta_3 \beta_2), \\ a_3 a_6 - a_2 a_7 &= \frac{1}{4} \alpha_1^2 \gamma_3^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (\beta_1 \beta_4 - \beta_3 \beta_2), \\ a_3 a_5 - a_1 a_7 &= -\frac{1}{4} \alpha_1^2 \delta_3^2 (\beta_1 \beta_4 - \beta_3 \beta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_2 a_4 - a_0 a_6 &= -\frac{1}{4} \alpha_1^2 \delta_1^2 (\beta_1 \beta_4 - \beta_3 \beta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_8 a_{11} - a_9 a_{10} &= -\frac{1}{4} \alpha_3^2 \beta_1^2 (-\delta_3 \delta_2 + \delta_1 \delta_4) (-\gamma_2 \gamma_3 + \gamma_4 \gamma_1), \\ a_{12} a_{15} - a_{13} a_{14} &= \frac{1}{4} \alpha_3^2 \beta_3^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_9 a_{12} - a_8 a_{13} &= \frac{1}{4} \alpha_3^2 \gamma_1^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (\beta_1 \beta_4 - \beta_3 \beta_2), \\ a_{11} a_{14} - a_{10} a_{15} &= \frac{1}{4} \alpha_3^2 \gamma_3^2 (\delta_1 \delta_4 - \delta_3 \delta_2) (\beta_1 \beta_4 - \beta_3 \beta_2), \\ a_{11} a_{13} - a_9 a_{15} &= -\frac{1}{4} \alpha_3^2 \delta_3^2 (\beta_1 \beta_4 - \beta_3 \beta_2) (-\gamma_4 \gamma_1 + \gamma_2 \gamma_3), \\ a_{10} a_{12} - a_8 a_{14} &= \frac{1}{4} \alpha_3^2 \delta_1^2 (\beta_1 \beta_4 - \beta_2 \beta_3) (\gamma_4 \gamma_1 - \gamma_2 \gamma_3). \end{aligned}$$

From the above equations, we conclude the equalities of this criterion.

Next we argue that the inequalities hold. Given α, β, γ and δ are invertible. Assume that $a_0 a_3 = a_1 a_2$, $a_5 a_6 = a_4 a_7$, $a_8 a_{11} = a_9 a_{10}$ and $a_{13} a_{14} = a_{12} a_{15}$. Then we obtain $\alpha_1 \beta_1 = 0$, $\alpha_1 \beta_3 = 0$, $\alpha_3 \beta_1 = 0$ and $\alpha_3 \beta_3 = 0$. Intuitively $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$ results in $\beta_1 = \beta_3 = 0$. Therefore these inequalities hold.

References

- [1] C. H. Bennett et al, quant-ph/9908073.
- [2] C. H. Bennett et al, Phys. Rev. A 63, 0123072001.
- [3] A. Acín et al., quant-ph/0003050.
- [4] W. Dür, G. Vidal and J.I. Cirac, Phys. Rev. A. 62 (2000)062314.
- [5] A. Acín, E. Jane, W. Dür and G. Vidal, Phys. Rev. Lett. 85, 4811 (2000).
- [6] H.J. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001).
- [7] H.K. LO and S. Popescu, Phys. Rev. A. 63 02230 (2001).
- [8] F. Verstraete, J. Dehaene and B. De Moor, Phys. Rev. A. 65, 032308 (2002).
- [9] F. Verstraete, J. Dehaene, B. De Moor and H. Verschelde Phys. Rev. A. 65, 052112 (2002).

- [10] Akimasa Miyake, quant-ph/0401023.
- [11] F.Pan et al., Phys. Lett. A 336, 384(2005).
- [12] A. Sampera, R. Tarrach, G. Vidal, Phys. Rev. A. 58 (1998)826-830.
- [13] D. Li et al., the necessary and sufficient conditions for separability for multipartite pure states, submitted.